Analytic expression for the structure factor and for the moment-generating function of fractal sets and multifractal measures



1997 J. Phys. A: Math. Gen. 30 4293

(http://iopscience.iop.org/0305-4470/30/12/017)

J. Phys. A: Math. Gen. **30** (1997) 4293–4312. Printed in the UK PII: S0305-4470(97)78331-8

# Analytic expression for the structure factor and for the moment-generating function of fractal sets and multifractal measures

Massimiliano Giona†

Centro Interuniversitario sui Sistemi Disordinati e sui Frattali nell'Ingegneria Chimica c/o Dip. Ingegneria Chimica Universitá di Roma 'La Sapienza', via Eudossiana 18, 00184 Roma, Italy

Received 2 October 1996, in final form 21 February 1997

**Abstract.** By applying the definition of Markov operator for iterated function systems with probability, an analytic expression for the structure factor and for the moment-generating function of a class of fractal sets is obtained in the presence of arbitrary multifractal distributions associated with affine iterated function systems. The properties of the structure factor of fractal sets are analysed in detail. The results developed for the structure factors have interesting implications as regards to the theory of Laplace transforms of singular non-atomic measures. Some physical applications of the theory, related to fractal and heterogeneous systems in the case of adsorption and relaxation, are briefly addressed. Integral transform theory is also applied to solve the Dirichlet problem for the Laplace equation on the circle in the presence of singular non-atomic boundary conditions.

## 1. Introduction

The analysis of the structure factor of fractal objects has important implications in the theory of diffraction and scattering [1, 2]. Scattering experiments are a powerful tool for analysing the fractal properties of aggregates, aerogels [3–5], suspensions [6, 7] and the surface structure of porous materials [8].

By applying renormalization to density, a recursive relation for the structure factor of $n$-order prefractals can be obtained [2, 9] for many fractal structures. For finitely ramified structures, the structure factor can easily be obtained by renormalization, in much the same way as for Green-function renormalization of wave-like excitations [10] and transport [11].

In this work, a rigorous analysis of the properties of the structure factor of generic self-affine sets is developed by using the theory of iterated function systems with probability (IFSP) [12, 13], and by applying the definition of the associated Markov operator. In this framework, the structure factor can be regarded as the Fourier transform of the unique invariant measure associated with the IFSP. As a consequence, the results obtained are directly applicable to any kind of multifractal distribution generated by means of affine maps. In particular, for a large class of IFSP labelled unimodular (see section 2 for the definition), a closed-form expression is derived for the structure factor.

The approach followed throughout this paper in order to obtain a functional equation for the structure factor is conceptually analogous to the analysis developed by Bessis and

† Permanent address: Dipartimento di Ingegneria Chimica Universitá di Cagliari, piazza d'Armi, 09123 Cagliari, Italy. E-mail address: max@giona2.ing.uniroma1.it

0305-4470/97/124293+20$19.50  

Demko [14], Abenda *et al* [15], Abenda and Turchetti [16], Forte and Vrscay [17] to derive a recursive equation for the moment hierarchy associated with the invariant measure of an IFSP.

The paper is organized as follows. In section 2 the fundamental definitions and results of the theory of IFSP are briefly reviewed as they are used throughout the article. A functional recursion for the structure factor of affine sets (affine IFSP) is then derived and a closed-form expression for unimodular sets is obtained in terms of a convergent infinite product. The scaling properties of the structure factor and its differentiability as a function of its argument and of the probability parameters characterizing the invariant measure are analysed and some examples are discussed. Section 6 analyses the Laplace transforms and presents some mathematical properties of the convolution of multifractal measures. Section 7 briefly discusses some physical implications of the theory in optics and in the characterization of heterogeneous materials (adsorption, relaxation). Finally, section 8 addresses the solution of boundary-value problems in the presence of singular non-atomic distributions†.

## 2. Iterated function systems with probability

This section briefly reviews the basic definitions in the theory of iterated function systems. For further details see [18, 12, 13].

Let $\mathcal{S}_w = \{\boldsymbol{w}_h(\boldsymbol{x})\}_{h=1}^N$ be a system of contraction maps on a metric space $(X, d)$. The base space $X$ is a compact subspace of $\mathbb{R}^n$ and $d$ a distance function defined on it. The system $\mathcal{S}_w$ is called an iterated function system (IFS).

Following Hutchinson [19], a set $\mathcal{C}$ is said to be *fractal* if there exists an IFS $\mathcal{S}_w$ such that

$$\mathcal{C} = \bigcup_{h=1}^{N} \boldsymbol{w}_h(\mathcal{C}). \tag{1}$$

If all $\boldsymbol{w}_h$ are affine, i.e.

$$\boldsymbol{w}_h(x) = \mathbf{A}_h \boldsymbol{x} + \boldsymbol{b}_h \qquad k = 1, \ldots, N \tag{2}$$

$\mathbf{A}_h$ being $n \times n$ matrices and $\boldsymbol{b}_h$ being $n$-dimensional vectors, the set $\mathcal{C}$, defined by the Hutchinson relation, equation (1), is said to be self-affine. In particular, if all $\boldsymbol{w}_h$ reduce to similitudes with scaling factors $a_h$, $(0 < a_h < 1)$, $\mathcal{C}$ is said to be self-similar.

We shall add another definition for further use. An IFS, and its limit set $\mathcal{C}$, are said to be *unimodular* if all the matrices $\mathbf{A}_h$ are equal, i.e. if all the maps $\boldsymbol{w}_h$ are characterized by a unique matrix $\mathbf{A}$ (modulus of the transformations) with different translation vectors.

Let $\Pi = \{p_h\}_{h=1}^N$ be a system of probability weights, $p_h > 0$, $\sum_{h=1}^N p_h = 1$. The pair $\{\mathcal{S}_w, \Pi\} = \{\boldsymbol{w}_h, p_h\}_{h=1}^N$ is defined as an IFSP.

In the set of probability measures $\mathcal{M}(X)$ on the $\sigma$-algebra of Borel subsets of $X$, the Markov operator $M : \mathcal{M}(X) \to \mathcal{M}(X)$ associated with the IFSP $\{\boldsymbol{w}_h, p_h\}_{h=1}^N$ is defined by the equation

$$M[\mu] = \sum_{h=1}^{N} p_h \mu \circ \boldsymbol{w}_h^{-1} \tag{3}$$

where $\circ$ indicates the composition operator. The space $(\mathcal{M}, d_H)$, equipped with the Hutchinson metrics $d_H$ [18], is a complete metric space and the Markov operator turns

† A probabilistic density $\rho(x)$ is said to be atomic if it is formed by a countable superposition of impulses, i.e. $\rho(x) = \sum_{h=1}^{\infty} \pi_h \delta(x - x_h)$, $\sum_{h=1}^{\infty} \pi_h = 1$.

out to be a contraction in it. Consequently, a unique invariant measure $\mu^*$ exists of the IFSP, which is a fixed point of the Markov operator

$$M[\mu^*] = \mu^*. \tag{4}$$

Let $\mu$ be a probability measure, $\nu = M[\mu]$, and $f : X \to C$ a generic continuous function attaining complex values. Then,

$$\int_X f(x)\,\mathrm{d}\nu\,(\boldsymbol{x}) = \int_X f(\boldsymbol{x})\,\mathrm{d}\,(M[\mu])(\boldsymbol{x}) = \sum_{h=1}^{N} p_h \int_X f \circ \boldsymbol{w}_h(\boldsymbol{x})\,\mathrm{d}\mu\,(\boldsymbol{x}). \tag{5}$$

In particular, if the probability measure $\mu$ in equation (5) is the invariant measure $\mu^*$, then $\nu = M[\mu^*] = \mu^*$, and equation (5) reduces to

$$\int_X f(\boldsymbol{x})\,\mathrm{d}\mu^*\,(\boldsymbol{x}) = \int_{\mathcal{C}} f(\boldsymbol{x})\,\mathrm{d}\mu^*\,(\boldsymbol{x}) = \sum_{h=1}^{N} p_h \int_{\mathcal{C}} f \circ \boldsymbol{w}_h(\boldsymbol{x})\,\mathrm{d}\mu^*\,(\boldsymbol{x}). \tag{6}$$

If the invariant measure $\mu^*$ coincides with a probabilistic Hausdorff measure of $\mathcal{C}$, we shall agree to call $\mu^*$ uniform. For self-similar non-overlapping (also called just-touching) IFSP [18], for which each matrix $\mathbf{A}_h$ is characterized by a scaling factor $a_h$, this occurs if $p_h = (a_h)^{d_H}$, $h = 1, \ldots, N$, where $d_H$ is the Hausdorff dimension, $\sum_{h=1}^{N}(a_h)^{d_H} = 1$ [20].

IFSP provides the simplest way to generate fractal sets $\mathcal{C}$, and multifractal measures $\mu^*$. It is therefore convenient to adopt the following definition of the structure factor $S(\boldsymbol{k})$ of the set $\mathcal{C}$ on which the measure (distribution function) $\mu^*$ is defined:

$$S(\boldsymbol{k}) = \int_{\mathcal{C}} \exp\left(\mathrm{i}\langle \boldsymbol{k}, \boldsymbol{x}\rangle\right)\mathrm{d}\mu^*\,(\boldsymbol{x}) \tag{7}$$

where i is the imaginary unit, $\boldsymbol{k} \in \mathbb{R}^n$ and $\langle\cdot,\cdot\rangle$ indicates the inner product. Definition (7) can be regarded as the Fourier–Stieltjes transform associated with $\mu^*$, whose support is $\mathcal{C}$.

Alternatively, in the case of one-dimensional structures, which can be defined without loss of generality on $\mathcal{C} \subseteq [0, 1]$, the moment-generating function $G(s)$ of $\mu^*$ can be defined as

$$G(s) = \int_{\mathcal{C}} \exp\left(-sx\right)\mathrm{d}\mu^*\,(x) \tag{8}$$

where $s$ is a complex variable and $x$ a scalar variable. Equation (8) can be regarded as the Laplace–Stieltjes transform of $\mu^*$, as discussed in section 6.

## 3. Structure factor

Let us apply equation (6) to the exponential kernel $f(\boldsymbol{x}) = \exp\left(\mathrm{i}\langle \boldsymbol{k}, \boldsymbol{x}\rangle\right)$ which defines the structure factor† $S(\boldsymbol{k})$

$$\begin{aligned} S(\boldsymbol{k}) &= \int_{\mathcal{C}} \exp\left(\mathrm{i}\langle \boldsymbol{k}, \boldsymbol{x}\rangle\right)\mathrm{d}\mu^*\,(\boldsymbol{x}) \\ &= \sum_{h=1}^{N} p_h \int_{\mathcal{C}} \exp\left(\mathrm{i}\langle \boldsymbol{k}, \mathbf{A}_h\boldsymbol{x} + \boldsymbol{b}_h\rangle\right)\mathrm{d}\mu^*\,(\boldsymbol{x}) \\ &= \sum_{h=1}^{N} p_h \exp\left(\mathrm{i}\langle \boldsymbol{k}, \boldsymbol{b}_h\rangle\right)\int_{\mathcal{C}} \exp\left(\mathrm{i}\langle \mathbf{A}_h^t\boldsymbol{k}, \boldsymbol{x}\rangle\right)\mathrm{d}\mu^*\,(\boldsymbol{x}) \end{aligned} \tag{9}$$

† Although other, slighlty different definitions for the structure factor may be assumed, equation (9) is analogous to the definition given in [9, 2].

where $\mathbf{A}_h^t$ is the transpose of $\mathbf{A}_h$. Therefore,

$$S(\boldsymbol{k}) = \sum_{h=1}^{N} p_h \exp(\mathrm{i}\langle \boldsymbol{k}, \boldsymbol{b}_h\rangle) S(\mathbf{A}_h^t \boldsymbol{k}) = \mathcal{F}[S(\boldsymbol{k})]. \tag{10}$$

Equation (10) is the functional equation for the structure factor. It is important to observe that equation (10) is also the spectral representation of equation (3) for the Markov operator of affine IFSP.

In the case of unimodular structures, equation (10) reduces to

$$S(\boldsymbol{k}) = \left[ \sum_{h=1}^{N} p_h \exp(\mathrm{i}\langle \boldsymbol{k}, \boldsymbol{b}_h\rangle) \right] S(\mathbf{A}^t \boldsymbol{k}). \tag{11}$$

The closure condition, for $\boldsymbol{k} = 0$, is of course

$$S(0) = 1. \tag{12}$$

Let us consider the functional $\mathcal{F}$ defined in the space of complex bounded functions $B(\mathbb{R}^n)$, equipped with the uniform metrics $d_\infty$,

$$d_\infty(S_1, S_2) = \sup_{\boldsymbol{k}} \|S_1(\boldsymbol{k}) - S_2(\boldsymbol{k})\| \qquad S_1, S_2 \in B(\mathbb{R}^n) \tag{13}$$

where $\|\cdot\|$ indicates the modulus of a complex number. If $S_1(\boldsymbol{k})$ belongs to $B(\mathbb{R}^n)$, then $S_2(\boldsymbol{k}) = \mathcal{F}[S_1(\boldsymbol{k})]$ belongs to $B(\mathbb{R}^n)$. Moreover, if $S_1, S_2 \in B(\mathbb{R}^n)$, then

$$d_\infty(\mathcal{F}[S_2], \mathcal{F}[S_1]) = \sup_{\boldsymbol{k}} \left\| \sum_{h=1}^{N} p_h \exp(\mathrm{i}\langle \boldsymbol{k}, \boldsymbol{x}\rangle)[S_2(\mathbf{A}_h^t \boldsymbol{k}) - S_1(\mathbf{A}_h^t \boldsymbol{k})] \right\| \leqslant d_\infty(S_2, S_1). \tag{14}$$

The functional $\mathcal{F}$ defined by equation (10) is not a contraction, due to its behaviour for $\boldsymbol{k} = 0$. Nevertheless, its iteratives $S_{n+1} = \mathcal{F}[S_n]$ converge uniformly in any bounded $n$-dimensional interval $-K_{\max} \leqslant k_i \leqslant K_{\max}$, $i = 1, \ldots, n$.

Let us consider the particular case of unimodular IFSP, equation (11), and the sequence of functions

$$S_{n+1}(\boldsymbol{k}) = \left[ \sum_{h=1}^{N} p_h \exp(\mathrm{i}\langle \boldsymbol{k}, \boldsymbol{b}_h\rangle) \right] S_n(\mathbf{A}^t \boldsymbol{k}) \qquad S_0(\boldsymbol{k}) = 1. \tag{15}$$

The initial function $S_0 = 1$ of this iterative scheme has been chosen in order to satisfy the closure condition, equation (12). It should be noted, however, that any bounded function $f(\boldsymbol{k})$ with $f(0) = 1$ could equally well have been chosen as the starting point of this iteration. The limit function $S_\infty(\boldsymbol{k})$ associated with the iterative recursion equation (15) is

$$S_\infty(\boldsymbol{k}) = \prod_{n=0}^{\infty} \left[ \sum_{h=1}^{N} p_h \exp(\mathrm{i}\langle (\mathbf{A}^t)^n \boldsymbol{k}, \boldsymbol{b}_h\rangle) \right]. \tag{16}$$

It is a straightforward matter to check that the function $S_\infty$(k) defined by equation (16) satisfies the functional equation (11) and the closure condition (12). We may therefore conclude that equation (16) is the closed-form expression for the structure factor of unimodular IFSP.

The results obtained in this section are summarized by the following proposition.

*Proposition 1.* The static structure factor associated with unimodular affine sets $\mathcal{C}$ generated by means of the IFSP $\{\mathbf{A}\boldsymbol{x} + \boldsymbol{b}_h, p_h\}_{h=1}^N$ is given by equation (16).

Moreover, starting from $S_0(\boldsymbol{k}) = 1$, the iteratives $S_n(\boldsymbol{k})$ of the recursion relation equation (15),

$$S_n(\boldsymbol{k}) = \prod_{j=0}^{n}\left[\sum_{h=1}^{N} p_h \exp(i\langle(\mathbf{A}^t)^j\boldsymbol{k}, \boldsymbol{b}_h\rangle)\right]. \tag{17}$$

converge uniformly in any bounded set of $\boldsymbol{k}$.

The latter statement provides a simple way to compute $S(\boldsymbol{k})$ for unimodular IFSP within arbitrary limits of accuracy.

It is interesting to observe that Zygmund [21] studied the Fourier-Stieltjes coefficients in the particular case of one-dimensional Cantor sets by following another approach based on identifying the structure of the set at iteration $n$ and then performing the limit for $n \to \infty$. The class of Cantor sets considered by Zygmund (see [21, p 195]) is defined on $[0, 2\pi]$, and is generated by means of the two-map IFSP, $w_1(x) = ax$, $w_2(x) = ax + 2\pi(1-a)$, with $p_1 = p_2 = \frac{1}{2}$, and therefore is unimodular. In particular, Zygmund considered the Fourier–Stieltjes coefficients $c_n = (2\pi)^{-1}\int_0^{2\pi}\exp(-\mathrm{i}nx)\,\mathrm{d}\mu^*(x)$. By applying equation (16) to this IFSP, it follows that

$$\begin{aligned} c_n &= \frac{1}{2\pi}\prod_{h=0}^{\infty}\left[\frac{1+\exp(-\mathrm{i}2\pi n a^h(1-a))}{2}\right] \\ &= \frac{1}{2\pi}\prod_{h=0}^{\infty}\exp(-\mathrm{i}n\pi a^h(1-a))\prod_{h=0}^{\infty}\cos(n\pi a^h(1-a)) \\ &= \frac{(-1)^n}{2\pi}\prod_{h=0}^{\infty}\cos(n\pi a^h(1-a)) \end{aligned} \tag{18}$$

which is exactly the expression for the Fourier–Stieltjes coefficients obtained by Zygmund.

Let us now consider equation (10) in the case of one-dimensional IFSP, $w_h(x) = a_h x + b_h$. The moment hierarchy $\{m_h\}$ associated with $\mu^*$ is given by

$$m_n = \int_{\mathcal{C}} x^n\,\mathrm{d}\mu^*(x) = (\mathrm{i})^{-n}\left.\frac{\mathrm{d}^n S(k)}{\mathrm{d}k^n}\right|_{k=0}. \tag{19}$$

By applying equation (10) it follows that

$$\begin{aligned} m_n &= (\mathrm{i})^{-n}\sum_{h=1}^{N} p_h \frac{\mathrm{d}^n}{\mathrm{d}k^n}[\exp(\mathrm{i}kb_h)S(a_h k)]_{k=0} \\ &= (\mathrm{i})^{-n}\sum_{h=1}^{N} p_h \sum_{j=0}^{n}\binom{n}{j}(\mathrm{i})^{n-j} b_h^{n-j} a_h^j \left.\frac{\mathrm{d}^j S(z)}{\mathrm{d}z^j}\right|_{z=0} \\ &= \sum_{h=1}^{N} p_h \sum_{j=0}^{n}\binom{n}{j} b_h^{n-j} a_h^j m_j \end{aligned} \tag{20}$$

from which the recursion scheme for the moment hierarchy reported in [14, 17] follows:

$$\left[1 - \sum_{h=1}^{N} p_h a_h^n\right] m_n = \sum_{h=1}^{N} p_h \sum_{j=0}^{n-1}\binom{n}{j} b_h^{n-j} a_h^j m_j. \tag{21}$$

A similar result can be obtained in higher dimensions by applying equation (10) in matrix form.

## 4. Scaling properties and differentiability

In this section we analyse the differentiability and scaling properties of the structure factor. In order to keep the notation as simple as possible, one-dimensional IFSP are considered.

A simple but important property of the structure factor of fractal sets possessing uniform or multifractal measures is its differentiability with respect to the argument $k$. Indeed, it is easy to see that the derivative of $S(k)$ exists for all $k$ and is given by

$$\frac{\mathrm{d}S(k)}{\mathrm{d}k} = \mathrm{i}\int_{\mathcal{C}} x\exp(\mathrm{i}kx)\,\mathrm{d}\mu^*(x) = \mathrm{i}D(k). \tag{22}$$

The functional equation for $D(k)$ in the case of unimodular IFSP can be derived by applying equation (6) to the function $f(x) = x\exp(\mathrm{i}kx)$,

$$D(k) = a\sum_{h=1}^{N} p_h \exp(\mathrm{i}kb_h)D(ak) + \sum_{h=1}^{N} p_h b_h \exp(\mathrm{i}kb_h)S(ak). \tag{23}$$

Instead of solving the functional equation (22) directly, its solution can be obtained by observing that the logarithm of $S(k)$ is given by

$$\log S(k) = \sum_{n=0}^{\infty} \log\left[\sum_{h=1}^{N} p_h \exp(\mathrm{i}a^n b_h k)\right]. \tag{24}$$

Therefore,

$$\frac{\mathrm{d}S(k)}{\mathrm{d}k} = S(k)\frac{\mathrm{d}\log S(k)}{\mathrm{d}k} = \mathrm{i}S(k)\sum_{n=0}^{\infty}\frac{P_n(k)}{T_n(k)}a^n \tag{25}$$

where

$$P_n(k) = \sum_{h=1}^{N} p_h b_h \exp(\mathrm{i}a^n b_h k) \qquad T_n(k) = \sum_{h=1}^{N} p_h \exp(\mathrm{i}a^n b_h k). \tag{26}$$

The series on the right-hand side of equation (26) is convergent since $a < 1$ and $\|P_n(k)/T_n(k)\| \leqslant \sum_{h=1}^{N} p_h|b_h|$.

For example, let us consider the Cantor middle-third set. The IFSP generating this structure with a uniform invariant measure is unimodular and given by the two maps $w_1(x) = x/3$, $w_2(x) = x/3 + \frac{2}{3}$ with $p_1 = p_2 = \frac{1}{2}$. Figure 1($a$) shows the modulus $I(k) = \|S(k)\|$ of the structure factor for this set and figure 1($b$) shows the derivative of the real part in a given interval of $k$.

Let us now consider the scaling behaviour of the moment-generating function of one-dimensional affine IFSP. With no loss of generality, let $\mathcal{C} \subseteq [0, 1]$. Moreover, let us order the maps $w_h$ for increasing values of $b_h$, i.e. $b_h < b_{h+1}$, and let us assume $b_1 = 0$.

The moment-generating function $G(s)$ is defined in the right half plane of the complex variable $s$, $\mathrm{Re}[s] \geqslant 0$ and satisfies a functional equation analogous to equation (10),

$$G(s) = \sum_{h=1}^{N} p_h \exp(-sb_h)G(a_h s). \tag{27}$$

Figure 2 shows the behaviour of $G(s)$ versus $s$ for the two-map IFSP, $w_1(x) = x/2$ with probability $p_1 = p$, $w_2(x) = x/2 + \frac{1}{2}$ with probability $p_2 = 1 - p$. This IFSP generates a multifractal binomial measure on the unit interval whose properties depend on $p$. A power-law behaviour of $G(s)$ can be observed for large $s$ with an exponent $\beta$ which depends on $p$,

$$G(s) = As^{-\beta} + \mathrm{o}(s^{-\beta}). \tag{28}$$

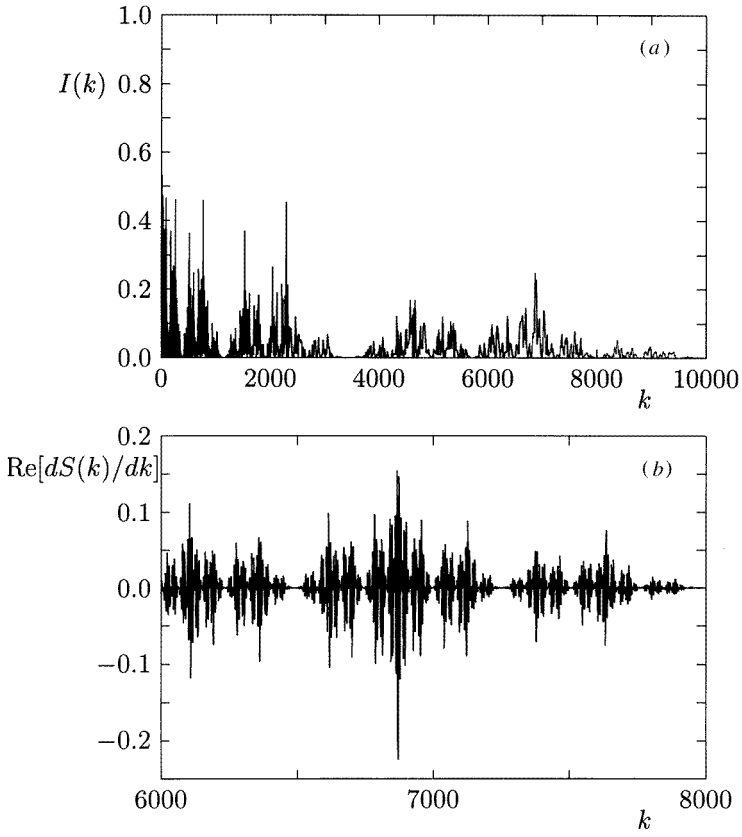


**Figure 1.** (*a*) $I(k) = \|S(k)\|$ for the Cantor middle-third set; (*b*) Snapshot of the derivative of the real part of $S(k)$.

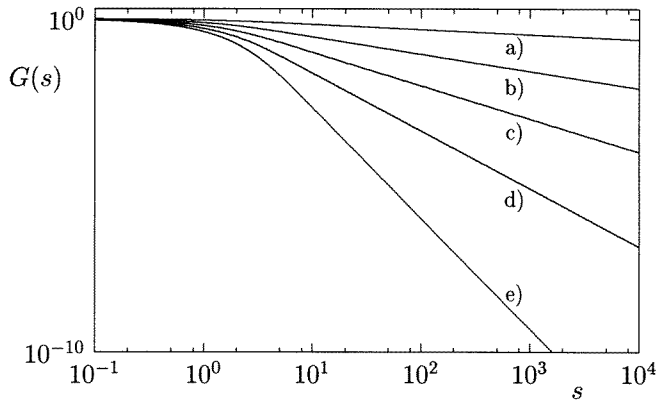


**Figure 2.** Moment-generating function $G(s)$ for a binomial measure on the unit interval generated by a two-map similar IFSP for different values of the probability $p_1 = p$. (*a*) $p = 0.9$; (*b*) $p = 0.7$; (*c*) $p = 0.5$ for which $G(s) = (1 - \exp(-s))/s$; (*d*) $p = 0.3$; (*e*) $p = 0.1$.

By substituting equation (28) into equation (27) we have asymptotically:

$$As^{-\beta} = Ap_1 a_1^{-\beta} s^{-\beta} + \mathrm{o}(s^{-\beta}\exp(-sb_2)). \tag{29}$$

Therefore, for $s \to \infty$, $1 = p_1 a_1^{-\beta}$, which yields

$$\beta = \frac{\log p_1}{\log a_1}. \tag{30}$$

In the case of the structure factor, the scaling exponent may be different from equation (30). In order to highlight this point, let us consider a two-map IFSP, $w_1(x) = ax$ with probability $p$, $w_2(x) = ax + (1-a)$ with probability $1-p$. Application of equation (16) gives the following expression for the modulus $I(k;p) = \|S(k;p)\|$ for a value $p$ of the probability weight,

$$I(k;p) = \prod_{n=0}^{\infty}[p^2 + (1-p)^2 + 2p(1-p)\cos(k(1-a)a^n)]^{1/2}. \tag{31}$$

Therefore,

$$I(k;p) = I(k;1-p) \tag{32}$$

i.e. the intensity $I(k;p)$ is symmetrical with respect to $p$ around $p = 0.5$. It is thus easy to see that equation (30) does not apply. The reason for this different scaling behaviour is that while $\exp(-sb_h) = \mathrm{o}(s^{-\beta})$ for $b_h > 0$, this is no longer true for the oscillating term $\exp(\mathrm{i}kb_h)$. In the case of one-dimensional unimodular IFSP, the following scaling behaviour has been observed for $I(k)$:

$$I(k) \sim k^{-\beta_k} \qquad \beta_k = \frac{\log p_{\max}}{\log a} \tag{33}$$

where $p_{\max} = \max_{h=1,\ldots,N}\{p_h\}$.

The situation is more complex in higher dimensions, for which anisotropies may appear in the scaling behaviour of the structure factor in all cases where the matrix $\mathbf{A}$ is not a similitude but possesses different scaling factors along the different Cartesian coordinates (self-affine structures).

## 5. Examples

This section discusses some examples related to two-dimensional structures. Let us consider the case of a unimodular IFSP in the plane. The matrix $\mathbf{A}$ is assumed to be the result of a similar contraction by a factor of $a$ and of a rotation with angle $\theta$:

$$\mathbf{A} = a\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{34}$$

Therefore,

$$(\mathbf{A}^t)^n = a^n\begin{pmatrix} \cos(n\theta) & \sin(n\theta) \\ -\sin(n\theta) & \cos(n\theta) \end{pmatrix}. \tag{35}$$

Let us indicate with $k_i$, and $b_{h,i}$ the $i$th component of the vector $\boldsymbol{k}$, $\boldsymbol{b}_{h,i}$. Equation (16) reduces to

$$S(\boldsymbol{k}) = \prod_{n=0}^{\infty}\left[\sum_{h=1}^{N} p_h \exp(\mathrm{i}a^n\varphi_h(n,\boldsymbol{k}))\right] \tag{36}$$

where

$$\varphi_h(n, \boldsymbol{k}) = \cos(n\theta)[k_1 b_{h,1} + k_2 b_{h,2}] + \sin(n\theta)[k_2 b_{h,1} - k_1 b_{h,2}]. \tag{37}$$

For example, figure 3 shows the contour plot of the intensity $I(\boldsymbol{k}) = \|S(\boldsymbol{k})\|$ for the Sierpinski carpet and figure 4 the moment-generating function $G(\boldsymbol{s}) = G(s_1, s_2)$ as a function of $s_1$ for different orientations, i.e. along the lines $s_2 = \alpha s_1$. The IFSP generating the Sierpinski carpet with a uniform measure is unimodular and represented by eight maps with $p_i = \frac{1}{8}$ $(i = 1, \ldots, 8)$, $a = \frac{1}{3}$ and $\theta = 0$ (in equation (34)). The translation vectors are $\boldsymbol{b}_1 = (0, 0)^t$, $\boldsymbol{b}_2 = (0, \frac{1}{3})^t$, $\boldsymbol{b}_3 = (0, \frac{2}{3})^t$, $\boldsymbol{b}_4 = (\frac{1}{3}, 0)^t$, $\boldsymbol{b}_5 = (\frac{1}{3}, \frac{2}{3})^t$, $\boldsymbol{b}_6 = (\frac{2}{3}, 0)^t$, $\boldsymbol{b}_7 = (\frac{2}{3}, \frac{1}{3})^t$, $\boldsymbol{b}_8 = (\frac{2}{3}, \frac{2}{3})^t$. As can be observed, for $s_2 = 0$, i.e. along the two principal directions of the Cartesian reference system, the scaling exponent of the moment-generating function is given by $\beta = \log(\frac{3}{8})/\log(\frac{1}{3})$, while for all the other orientations $\beta = \log 8/\log 3 = d_H$. The apparently anomalous behaviour along the Cartesian axes can be understood by means of equation (30). Indeed, for $s_2 = 0$, $G(s_1, 0)$ corresponds to the moment-generating function associated with the projected measure $\mu_1(x_1)$, with $\mathrm{d}\mu_1^*(x_1) = \int_{x_2=-\infty}^{\infty} \mathrm{d}\mu^*(\boldsymbol{x})$ which corresponds to a three-map one-dimensional IFSP with

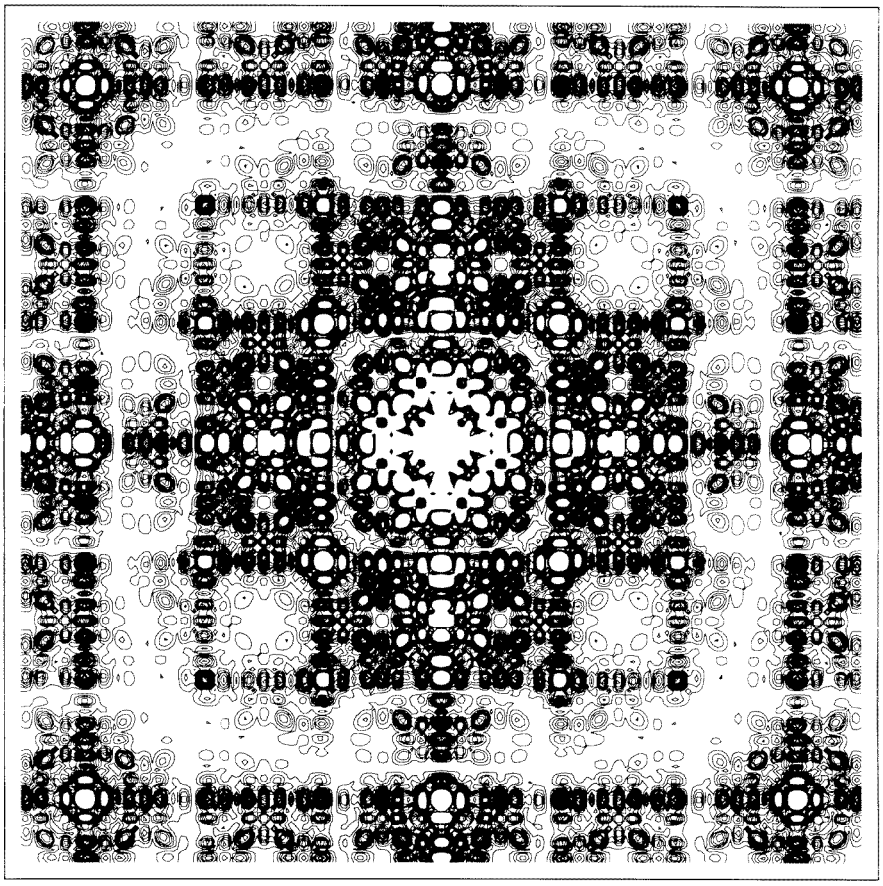

**Figure 3.** Contour plot of $I(\boldsymbol{k}) = I(k_1, k_2)$ for $k_i$ $(i = 1, 2)$ ranging in the interval $-200 \leqslant k_i \leqslant 200$ in the case of the Sierpinski carpet.

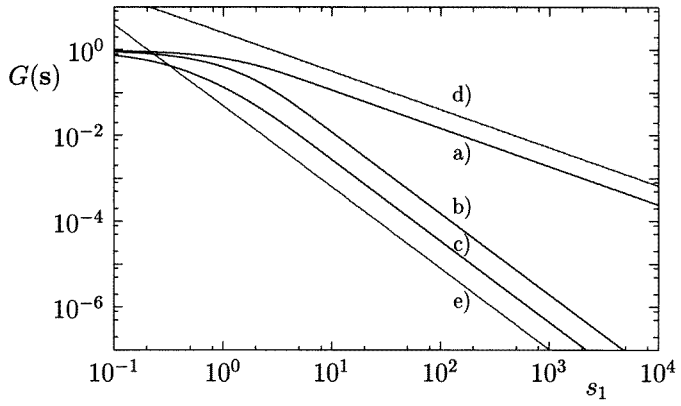


**Figure 4.** Moment-generating function $G(s)$ versus $s_1$ along different orientations, i.e. for $s_2 = \alpha s_1$ with $\alpha$ constant. (*a*) $\alpha = 0$; (*b*) $\alpha = 1$; (*c*) $\alpha = 5$; (*d*) $G \sim s_1^{-\beta}$ with $\beta = \log(\frac{8}{3})/\log 3$; (*e*) $G \sim s_1^{-d_H}$.

on $w_1(x) = x/3$, $p_1 = \frac{3}{8}$, $w_2(x) = x/3 + \frac{1}{3}$, $p_2 = \frac{2}{8}$, $w_3(x) = x/3 + \frac{2}{3}$, $p_3 = \frac{3}{8}$. The exponent $\beta$ is therefore given by equation (30) with $p_1 = \frac{3}{8}$, $a_1 = a = \frac{1}{3}$.

As a further example, figure 5 shows the contour plot of $I(\boldsymbol{k})$ for the Vicsek fractal (studied by Allain and Cloitre by applying renormalization). This set is generated by means of a five-map unimodular IFSP with $\mathbf{A}$ given by equation (34) with $a = \frac{1}{3}$, $\theta = 0$, and with $\boldsymbol{b}_1 = (\frac{1}{3}, 0)^t$, $\boldsymbol{b}_2 = (0, \frac{1}{3})^t$, $\boldsymbol{b}_3 = (\frac{1}{3}, \frac{1}{3})^t$, $\boldsymbol{b}_4 = (\frac{2}{3}, \frac{1}{3})^t$, $\boldsymbol{b}_5 = (\frac{1}{3}, \frac{2}{3})^t$. A uniform invariant measure is obtained for $p_i = \frac{1}{5}$ $(i = 1, \ldots, 5)$.

## 6. Laplace transforms of multifractal measures

A direct application of the theory presented above is related to the Laplace–Stieltjes transform since, as a consequence of proposition 1, it is possible to obtain a closed-form expression for the Laplace transform (moment-generating function) of multifractal measures. To this end, let us consider one-dimensional unimodular IFSP such that $\mathcal{C} \subseteq [0, 1]$, and let us define the Laplace–Stieltjes transform associated with $\mu^*$, $\mathcal{L}[\mathrm{d}\mu^*]$, as

$$\mathcal{L}[\mathrm{d}\mu^*(x)] = \int_{\mathcal{C}} \exp(-sx)\,\mathrm{d}\mu^*(x). \tag{38}$$

Taking into account the fact that the moment-generating function equation (8) can be viewed as the Laplace transform $\mathcal{L}[\mathrm{d}\mu^*]$, we have the following proposition.

*Proposition 2.* Let $\mu^*$ be the invariant measure associated with a one-dimensional unimodular IFSP $\{w_h, p_h\}_{h=1}^N$, $w_h = ax + b_h$. Then

$$\mathcal{L}[\mathrm{d}\mu^*(x)] = \prod_{n=0}^{\infty}\left[\sum_{h=1}^{N} p_h \exp(-a^n b_h s)\right]. \tag{39}$$

In particular, in the case of the unit interval equipped with a binomial measure, i.e. $w_1(x) = x/2$ with probability $p$, $w_2(x) = x/2 + \frac{1}{2}$ with probability $1 - p$, it follows that

$$\mathcal{L}[\mathrm{d}\mu^*(x)] = \prod_{n=0}^{\infty}\left[p + (1-p)\exp(-s/2^{n+1})\right] \tag{40}$$

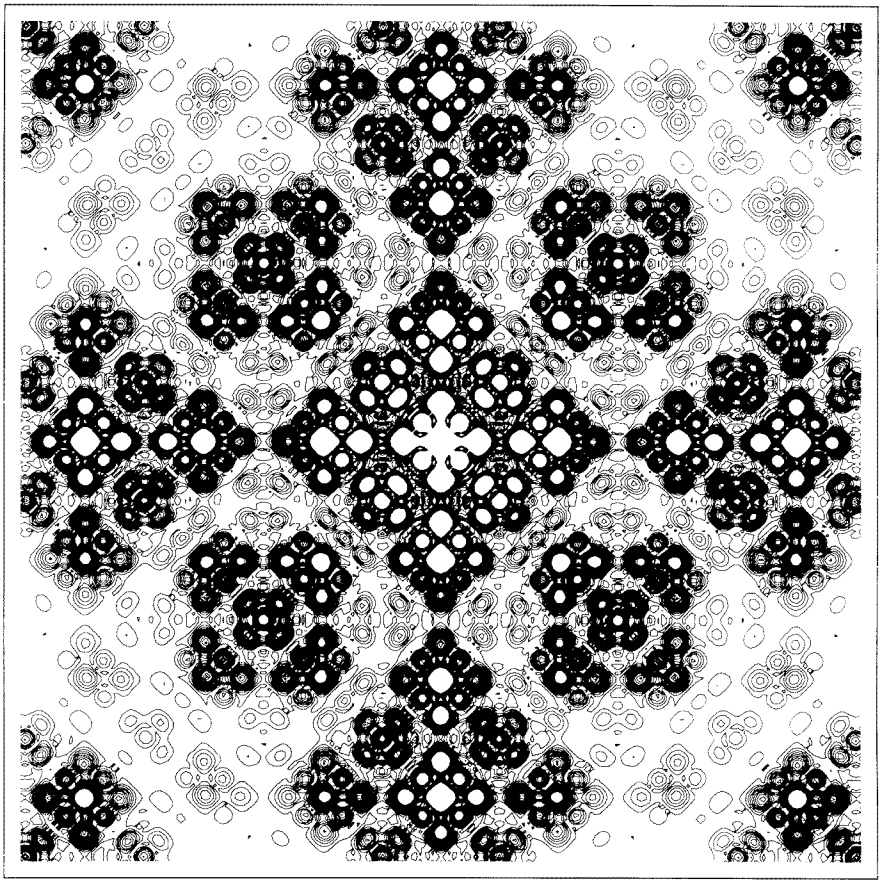

**Figure 5.** Contour plot of $I(\boldsymbol{k}) = I(k_1, k_2)$ for $k_i$ $(i = 1, 2)$ ranging in the interval $-200 \leqslant k_i \leqslant 200$ in the case of the Vicsek fractal.

and for Cantor sets generated by a two-map IFSP, $w_1(x) = ax$ with probability $p$, $w_2(x) = ax + (1 - a)$ with probability $1 - p$ $(a < \frac{1}{2})$:

$$\mathcal{L}[\mathrm{d}\mu^*(x)] = \prod_{n=0}^{\infty}[p + (1-p)\exp(-(1-a)a^n s)]. \tag{41}$$

The uniform convergency properties for the $n$-order approximations discussed in section 3, equation (17), hold for the infinite products defined in equations (39)–(41).

As a consequence of equations (39) and (40), the following identities (obtained by applying these equations to a uniform measure on the unit interval) hold:

$$\frac{1-\exp(-s)}{s} = \prod_{n=0}^{\infty}[\tfrac{1}{2} + \tfrac{1}{2}\exp(-s/2^{n+1})] \tag{42}$$

which is an application of equation (40) with $p = \frac{1}{2}$ and

$$\frac{1-\exp(-s)}{s} = \prod_{n=0}^{\infty}\left[\frac{1}{N}\frac{1-\exp(-s/N^n)}{1-\exp(-s/N^{n+1})}\right] \qquad \text{for } N = 1, 2, \ldots \tag{43}$$

which is an application of equation (39) with $p_h = 1/N$ and $b_h = (h-1)/N$.

Another interesting property is that the moment-generating function $G(s)$, equation (8), is differentiable with respect to the probability weights $p_h$.

As an example, let us take the moment-generating function of binomial measures on the unit interval and write $G(s) = G(s; p)$ to indicate explicitly the dependence of the moment-generating function on the parameter $p$. It can be shown that the derivative of $G(s; p)$ with respect to $p$ exists and is given by

$$\frac{\partial G(s;p)}{\partial p} = \prod_{n=0}^{\infty}[p + (1-p)\exp(-s/2^{n+1})] \sum_{h=0}^{\infty} \frac{1-\exp(-s/2^{h+1})}{p + (1-p)\exp(-s/2^{h+1})}. \tag{44}$$

The results derived for Laplace–Stieltjes transforms of multifractal measures may be applied to all the problems involving convolutions.

A typical physical example arises in the study of the vibrational properties of product lattices, $\mathcal{G} = \mathcal{G}_1 \otimes \mathcal{G}_2$ obtained as the Cartesian product of finitely ramified fractals [22]. The density of states associated with scalar vibrations of the product lattice $\mathcal{G}$ is the convolution of the densities of states of $\mathcal{G}_1$ and $\mathcal{G}_2$. For fractal structures, the density of states gives rise to singular non-atomic measures, as discussed extensively by Domany *et al* [23]. It is important to observe that product lattices play an important role in the dynamic theory of fractal structures as they represent an example of fractal lattices whose spectral dimension may exceed 2 [24]. For finitely ramified fractals, such as the Sierpinski gasket, Vicsek fractal etc, the spectral dimension is always bounded by 2.

Moreover, product lattices are in general infinitely ramified, and therefore represent a more realistic model of real topologically complex porous and capillary networks. A detailed discussion of the physical implications of product lattices has been developed by Schwalm and Schwalm [22]. Hilfer and Blumen [25] have studied the density of states of the Cartesian product of a Sierpinski gasket with a one-dimensional line. An application of the theory presented in [22] to random structures was discussed by Dominguez and Wiecko [26].

Apart from the spectral properties of fractal structures, convolutions enter into the solution of linear Volterra integral equations of classical mathematical physics [27].

Let us study some mathematical properties of the convolution of (multifractal or not) measures defined on the unit interval or on Cantor dusts.

Let $\mu_1^*$ and $\mu_2^*$ be the invariant measures associated with two unimodular IFSP $\{a^{(1)}x + b_h^{(1)}, p_h^{(1)}\}_{h=1}^{N_1}$, $\{a^{(2)}x + b_h^{(2)}, p_h^{(2)}\}_{h=1}^{N_2}$. The convolution of $\mu_1^*$ and $\mu_2^*$ generates a new measure $\mu$ defined by the relation

$$\mathrm{d}\mu\,(x) = \int \mathrm{d}\mu_1^*\,(z)\,\mathrm{d}\mu_2^*\,(x-z). \tag{45}$$

In terms of Laplace transforms, from equation (45) it obviously follows $G(s) = \int \exp(-sx)\,\mathrm{d}\mu\,(x) = G_1(s)G_2(s)$. By applying equation (39) to $G(s)$ one obtains

$$G(s) = \prod_{n=0}^{\infty}\left[\sum_{h=1}^{N_1} p_h^{(1)} \exp(-(a^{(1)})^n b_h^{(1)} s)\right]\left[\sum_{k=1}^{N_2} p_k^{(2)} \exp(-(a^{(2)})^n b_k^{(2)} s)\right]. \tag{46}$$

In the case $a_1 = a_2 = a$, equation (46) simplifies to

$$G(s) = \prod_{n=0}^{\infty}\left[\sum_{h=1}^{N_1}\sum_{k=1}^{N_2} p_h^{(1)} p_k^{(2)} \exp(-a^n(b_h^{(1)} + b_k^{(2)})s)\right]. \tag{47}$$

By comparing equation (47) with equation (39), it can be readily seen that equation (47) corresponds to the Laplace transform of the invariant measure of a $N_1 \times N_2$-map unimodular

IFSP characterized by the transformations $w_{h,k}(x) = ax + b_h^{(1)} + b_k^{(2)}$ ($h = 1, \ldots, N_1$, $k = 1, \ldots, N_2$) equipped with the probability weights $p_{h,k} = p_h^{(1)} p_k^{(2)}$.

This result allows us to identify completely the resulting measure $\mu$, and can be summarized by means of the following proposition.

*Proposition 3.* The convolution of two invariant meaures associated with self-similar unimodular IFSP possessing the same scaling factor $a$, is the invariant measure of the unimodular IFSP with $w_{h,k} = ax + b_{h,k}$, $h = 1, \ldots, N_1$, $k = 1, \ldots, N_2$, where $b_{h,k}$ is the sum of the translation terms $b_h^{(1)} + b_k^{(2)}$ and the probability weights are the product of probabilities $p_{h,k} = p_1^{(1)} p_2^{(2)}$ of the two IFSP.

Let us make some applications of this result. Let us consider the two-map IFSP $w_1(x) = x/2$, $w_2(x) = x/2 + \frac{1}{2}$ with $p_1 = p_2 = \frac{1}{2}$, whose invariant measure is uniform, $\rho^*(x) = 1$ on the unit interval.

The $n$-order convolution of $\rho^*(x)$ with itself generates a regular density $\rho^{(n,*)}(x)$ on $[0, n+1]$, whose restriction to the unit interval is given by

$$\rho^{(n,*)}(x) = \frac{x^n}{n!} \qquad x \in [0, 1]. \tag{48}$$

By applying the proposition stated above, it follows that the IFSP generating $\rho^{(n,*)}(x)$ is given by the $(n+2)$-map IFSP with

$$w_h^{(n)}(x) = \frac{x + h}{2} \qquad p_h^{(n)} = \binom{n+1}{h} \qquad h = 0, \ldots, n+1. \tag{49}$$

The set of IFSP given by equation (49) may be defined as the *polynomial IFSP basis on the interval* $[0, 1]$ since, due to equation (48), it generates power-law invariant densities. This example indicates that regular non-uniform distributions of polynomial shape can be obtained by means of simple similar overlapping IFSP.

As a further example let us consider the convolution of two invariant measures associated with the two-map IFSP generating the Cantor middle-third set, $w_1(x) = x/3$ with probability $p_1$, $w_2(x) = x/3 + \frac{2}{3}$ with probability $p_2$. The convolution gives rise to a measure defined on the interval $[0, 2]$ and associated with the three-map IFSP $w_1^c(x) = x/3$ with probability $p_1^2$, $w_2^c(x) = x/3 + \frac{2}{3}$ with probability $2p_1p_2$, $w_3^c(x) = x/3 + \frac{4}{3}$ with probability $p_2^2$. The generalized dimensions $D(q)$ associated with the corresponding invariant measure can be obtained in closed form since the IFSP is just-touching,

$$D(q) = \frac{\log(p_1^{2q} + (2p_1p_2)^q + p_2^{2q})}{(1-q)\log 3}. \tag{50}$$

This example indicates that multifractal measures may arise from the convolution of two non-multifractal measures (as for $p_1 = p_2 = \frac{1}{2}$ in the previous example) defined on Cantor dust supports.

## 7. Physical applications of the theory

This section furnishes an outline of some physical applications of the integral transform theory with particular reference to open problems and current lines of research.

### *7.1. Optical applications*

To quote Lohmann [28]: '*the Fourier transform is of such central significance to physical optics that everything that is somehow related to Fourier mathematics is likely to be important in the realm of optics*'.

The theory of Fourier transforms does in fact find a direct application in the theory of scattering from fractal structures. The results given in section 5 for model fractal structures agree with the experiments performed by Reed [9].

The integral transform theory may prove a particularly useful way of attaining a rigorous mathematical analysis of the physical theories for scattering from pore fractal structures developed by Bale and Schmidt [8]. The scaling theory of Bale and Schmidt [8] is still the subject of discussion and investigation, and the application of the methods discussed throughout this article to some model structures of pore fractals may help towards a better understanding of this issue.

In the field of optics, the analysis of the Fourier–Stieltjes transform can be applied to other linear integral transforms of physical interest, and in particular to fractional Fourier transforms [28, 29] and wavelet transforms [30, 31].

To simplify the notation, let us consider a one-dimensional example. In general, wavelet transforms may be defined as

$$\Psi(\boldsymbol{k}) = \int_C \psi(x;\boldsymbol{k})\,\mathrm{d}\mu^*(x) \tag{51}$$

where $\boldsymbol{k}$ is a set of parameters, and the wavelet function $\psi$ is generally supposed to be of zero mean for the wavelet transform to be invertible.

Wavelet transforms can be regarded as a mathematical microscope highlighting the local properties of the object by a suitable choice of the wavelet $\psi$. Wavelet analysis is currently applied to study the structural and textural features of physical fractals: diffusion-limited aggregates, depositional aggregates, etc [32].

Let us consider a typical wavelet, such as the complex Morlet wavelet [30], defined by the kernel $\psi(x;k,\lambda) = \exp(\mathrm{i}kx - \lambda x^2)$.

By applying equation (6) to this wavelet transform in the case of linear IFSP, it follows that

$$\Psi(k,\lambda) = \sum_{h=1}^{N} p_h \exp(\mathrm{i}kb_h - \lambda x^2)\Psi(ka_h + 2\mathrm{i}\lambda a_h b_h, \lambda a_h). \tag{52}$$

A closed-form expression for $\Psi(k,\lambda)$ has yet to be found, and the recursive relation equation (52) should be approached in a numerical way. This situation is analogous to the case of the Fourier–Stieltjes transform or of the Laplace–Stieltjes transform for non-unimodular IFSP. It is therefore useful to examine briefly the computational issues associated with the recursive relations such as equation (52), by considering, for example the case of the Fourier–Stieltjes transform.

In those cases for which a closed-form expression can be found (unimodular IFSP), the analysis developed in sections 3 and 4 yields arbitrarily accurate results (and the accuracy can be easily controlled since the approximants, equation (17) are uniformly convergent). Consequently the application of equation (17) is both conceptually and numerically superior to any kind of numerical algorithm (such as FFT).

In all other cases, the numerical algorithm consists of the iteration of a recursive

functional equation, which for the Fourier–Stieltjes transform takes the form

$$S_{n+1}(\boldsymbol{k}) = \sum_{h=1}^{N} p_h \exp(\mathrm{i}\langle \boldsymbol{k}, \boldsymbol{b}_h\rangle) S_n(\mathbf{A}_h^t \boldsymbol{k}) \qquad S_0(\boldsymbol{k}) = 1 \tag{53}$$

and analogously for the wavelet transform by applying equation (52) to $\{\Psi_n(k, \lambda)\}$ since $\Psi(0, 0) = 1$.

The implementation of equation (53) can be performed by means of a recursive algorithm with memory in which the values of $S_n(\boldsymbol{k})$ at the preceding iterations are stored in an associative array $s(n, \boldsymbol{k})$, which is an array labelled by means of real numbers [33] (see also [34]).

A comparison with classical routines such as FFT reveals the basic differences between the two methods. In FFT, data are sampled and the accuracy depends on the sampling interval. The recursive algorithm equation (53) is independent of sampling, and corresponds to a renormalization procedure. Indeed, the $n$th iteration of the algorithm corresponds approximately to the $N^n$ sampling point in the FFT routine. Moreover, since the sequence $\{S_{n+1}(\boldsymbol{k})\}$ is uniformly convergent, numerical accuracy can be easily controlled.

The implementation of equation (53) requires, however, a significant data segment for the storing of the associative array $s(n, \boldsymbol{k})$. The memory occupied can be reduced by introducing an approximation, consisting of assuming that $s(n, \boldsymbol{k}) = s(n, \boldsymbol{k}')$ for $k' \leqslant k + \varepsilon$, where $\varepsilon$ (usually $\varepsilon \simeq 10^{-8}$–$10^{-12}$) controls the rounding-off error.

The computer time needed to implement equation (53) depends on the nature of the entries of the matrices $\mathbf{A}_h^t$. If the entries of $\mathbf{A}_h^t$ are rational numbers, the algorithm is very efficient and the data storage reduces drastically.

Preliminary analyses show that the algorithm is slower than FFT (but not significantly so) for small $n$, but is definitely more convenient for large $n$, i.e. for accurate spectral analysis and at large wavelengths. To give an example, the computation of the Fourier spectrum over $M = 10^4$ data points in the range $k \in (0, 10^5)$ for an IFSP with 4–8 maps at iteration $n = 100$ (corresponding roughly to $4^{100}$–$8^{100}$ sampling points, i.e. beyond the applicability of any FFT algorithm based on sampling), with a rounding-off threshold of $\varepsilon = 10^{-10}$, takes less than 10 min of CPU time on a standard IBM RISC workstation. The algorithm is also particularly suitable for parallel implementation.

Another interesting application of integral transform theory is related to the determination of thermodynamic averages in crystal models. Bessis and Demko [14] showed that the invariant measures of IFSP can be used to approximate the vibrational density of states of face-centred cubic crystal with nearest-neighbouring central force constants. This problem has been considered by Wheeler and Gordon [35] with reference to the computation of the zero-point vibrational energy, defined as $E_o = (\frac{1}{2}) \int_0^1 \sqrt{x}\, \mathrm{d}\mu^*(x)$, where $\mathrm{d}\mu^*(x) = \rho(x)\, \mathrm{d}x$, and $\rho(x)$ is the normalized vibrational density of states. Bessis and Demko [14] reconstructed the density of states by means of a four-map affine IFSP by matching the first nine moments of the measure. Integral transform theory provides a simple way to compute the averages of continuous transcendental functions such as $f(x) = \sqrt{x}$, since $f(x)$ can be expanded in Fourier series and the Fourier coefficients can be obtained by applying the methods discussed in sections 3 and 4.

### *7.2. Characterization and dynamics of heterogeneous systems*

Integral transform theory finds application also in the study of physical properties of heterogeneous systems. Two examples are briefly addressed.

The first case refers to the study of adsorption isotherms on heterogeneous solids. In the chemical physics of surfaces, heterogeneity is usually referred to those structures (i.e. the overwhelming majority) which are characterized by a broad distribution $\mu^*(E)$ of adsorption energies $E$ [36]. Broad distributions of adsorption energies arise from local defects, dislocations, substitutional disorder, etc.

If $\theta_{\rm loc}(P, T; E)$ indicates the local adsorption isotherm corresponding to an adsorption energy $E$ ($P$ is the pressure, $T$ is the temperature), the global, and experimentally measurable, adsorption isotherm $\theta(P, T)$ is given by

$$\theta(P, T) = \int_{\mathcal{C}} \theta_{\rm loc}(P, T; E)\, {\rm d}\mu^*(E) \tag{54}$$

where $\mathcal{C}$ is the set of admissible adsorption energies.

It is important to observe that $\mu^*(E)$ is not a measurable quantity, and can be obtained in an indirect way by inverting equation (54), once an expression for the local isotherm is chosen. Smooth and differentiable distributions of adsorption energies are usually considered [36]. From the theoretical point of view, however, it is interesting to also analyse the influence of singular distributions, which may be a reasonable assumption for highly singular adsorbents.

As a local adsorption isotherm, a Langmuir expression may be chosen [37, 38]

$$\theta_{\rm loc}(P, T; E) = \frac{KP\exp(E/RT)}{1 + KP\exp(E/RT)} \tag{55}$$

where $R$ is the gas constant and $K$ corresponds to the low-pressure Henry constant. By inserting equation (55) into equation (54) it follows that

$$\theta(P, T) = \alpha \int_{\mathcal{C}} \frac{{\rm d}\mu^*(E)}{\exp(-\beta E) + \alpha} = \alpha h(\beta, 1, \alpha) \tag{56}$$

where $\alpha = KP$, $\beta = 1/RT$ and the function $h(\beta, \lambda, \alpha)$ is given by

$$h(\beta, \lambda, \alpha) = \int_{\mathcal{C}} \frac{{\rm d}\mu^*(E)}{\lambda\exp(-\beta E) + \alpha}. \tag{57}$$

In the case of singular measures generated by means of linear IFSP, it readily follows that

$$h(\beta, \lambda, \alpha) = \sum_{h=1}^{N} p_h h(\beta a_h, \lambda\exp(-\beta b_h), \alpha) \tag{58}$$

and the analysis discussed throughout this article applies also to this case.

Another example is given by the study of relaxation properties of complex systems. In many physical phenomenologies, such as stress relaxation in polymers and suspensions, dielectric relaxation, and thermoremnant magnetization in spin glasses, anomalous relaxation properties have been observed [39]. If $\phi(t)$ indicates the relaxation function (such as polarization current in dielectric relaxation), a basic approach is to consider $\phi(t)$ as the superposition of a large number of exponentially decaying modes $\exp(-kt)$, where $t$ is time and $k$ is the relaxation rate, i.e. the reciprocal of the characteristic modal relaxation time, so that

$$\phi(t) = \int_{\mathcal{C}} \exp(-kt)\, {\rm d}\mu^*(k) \tag{59}$$

where $\mu^*(k)$ is the distribution of relaxation times. Equation (59) corresponds to the Laplace–Stieltjes transform of $\mu^*(k)$. In the study of the relaxation properties of fractals and of complex structures, the hierarchical distribution of relaxation times may be taken

into consideration [40]. These distributions may be described by means of singular non-atomic measures. In the presence of such distributions, it follows from equations (59) and (30) that $\phi(t) \sim t^{-\beta}$, i.e. the relaxation function is characterized by a power-law decay. Power-law relaxations have been experimentally observed in dielectrics (see [41–45] and references therein) and in viscoelastic materials (see [46, 47] and references therein), and have been theoretically interpreted by means of constitutive equations of Riemann–Liouville type [48, 49].

There are in fact some analogies between transport phenomena in the presence of multifractal distributions of transport and rate coefficients, and constitutive (flux/concentration gradient) equations of Riemann–Liouville type, as recently observed in [50]. The study of the analogies between distributed models characterized by multifractal properties in transport and rate coefficients and mean-field models with Riemann–Liouville kernels is needed to be taken further since it is of value in the study of dynamic phenomena in complex and fractal systems in order to formulate theoretical models and approximate mean-field equations interpreting transport, reaction and relaxation phenomena.

## 8. PDE with singular boundary conditions

The theory of integral transforms developed above can be applied to solve basic linear equations of mathematical physics in the presence of singular non-atomic boundary conditions.

This section develops as a case study of the solution of the Dirichlet problem for the Laplace equation on the circle. This is the simplest, non-trivial problem for elliptic equations, which finds applications in the theory of steady-state transport phenomena, e.g. heat transfer [51].

Let us consider the Laplace equation,

$$\nabla^2 c = 0 \tag{60}$$

inside a circle of radius $R$, i.e. for $r < R$, $\theta \in [0, 2\pi]$ equipped with the boundary conditions of Dirichlet type

$$c(R, \theta) = c_o(\theta). \tag{61}$$

The solution of this equation is referred to as the Poisson formula

$$c(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} K(r, \theta - \psi) c_o(\psi)\, \mathrm{d}\psi \tag{62}$$

where the Poisson kernel $K(r, \theta - \psi)$ is given by [27]

$$K(r, \theta - \psi) = \frac{1 - (r/R)^2}{1 - 2(r/R)\cos(\theta) + (r/R)^2} = 1 + 2\sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n \cos[n(\theta - \psi)]. \tag{63}$$

By changing the variables, $\theta = 2\pi y$, $\psi = 2\pi\eta$ ($y$, $\eta \in [0, 1]$), $\xi = r/R$, $\widetilde{c}_0(y) = c_0(2\pi y)$, the solution of the Dirichlet problem reads as

$$c(\xi, y) = \int_0^1 \widetilde{c}_0(y)\, \mathrm{d}y + 2\sum_{n=1}^{\infty} \xi^n \int_0^1 \widetilde{c}_0(\eta) \cos[2\pi(y - \eta)]\, \mathrm{d}\eta. \tag{64}$$

Let us consider the case where $\widetilde{c}_0(y)\, \mathrm{d}y$ is proportional to the distribution of $\mathrm{d}\mu^*(y)$ on $[0, 1]$ associated with an IFSP $C_0\, \mathrm{d}\mu(y) = \widetilde{c}_0(y)\, \mathrm{d}y$, where $C_0$ is a positive constant. From equation (64) it follows that

$$c(\xi, y) = C_0 \left[ 1 + 2\sum_{n=1}^{\infty} \xi^n (\mathrm{Re}[S(2\pi n)] \cos(2\pi y) + \mathrm{Im}[S(2\pi n)] \sin(2\pi y)) \right] \tag{65}$$

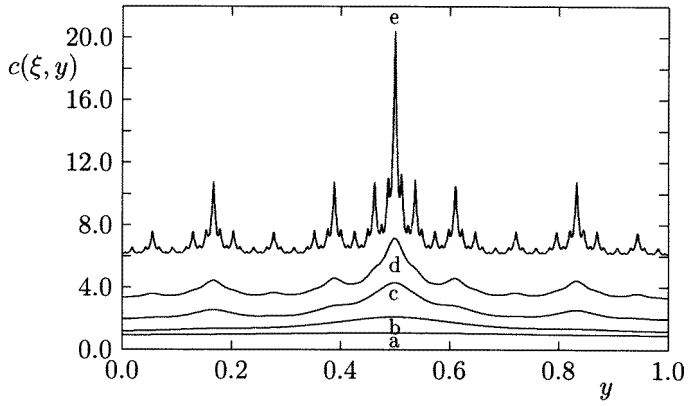


**Figure 6.** Behaviour of $c(\xi, y)$ versus $y$, equation (65) for different values of $\xi$ (from bottom up, $\xi = 0.1, 0.5, 0.8, 0.9, 0.99$). The $c(\xi, y)$-profiles have been translated along the vertical axis by a constant factor (i.e. the vertical scale is arbitrary) in order to achieve a better visual representation.

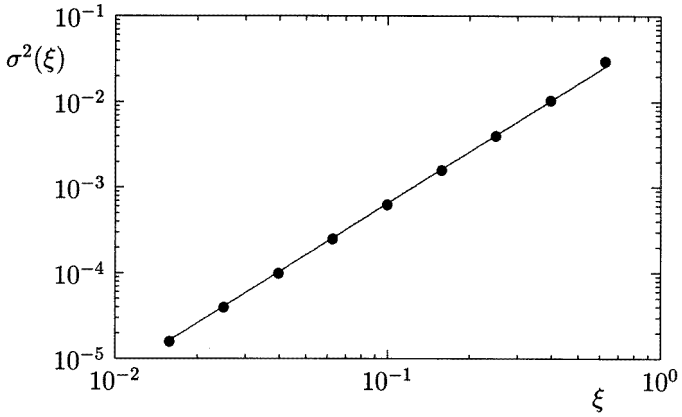


**Figure 7.** $\sigma^2(\xi)$ versus $\xi$. The dots represent the data obtained from the solution of the Laplace equation (65). The line is $\sigma^2(\xi) = A\xi^2$, with $A$ constant.

where $S(2\pi n)$, see equation (7), are the Fourier–Stieltjes coefficients of $\mu^*$.

Figure 6 shows the behaviour of $c(\xi, y)$ versus the angular coordinate $y$ for different values of $\xi$, where $\mu^*$ is the invariant measure associated with a three-map IFSP $w_h(x) = ax + b_h$ with $a = \frac{1}{3}$, $b_1 = 0$, $b_2 = \frac{1}{3}$, $b_3 = \frac{2}{3}$ for $p_1 = 0.2$, $p_2 = 0.6$, $p_3 = 0.2$. As can be expected, the solution $c(\xi, y)$ becomes rougher as $\xi$ increases up to the boundary. A measure of the roughening is given by the mean square deviation $\sigma^2(\xi) = \int_0^1 [c(\xi, y) - \langle c(\xi)\rangle]^2 \, \mathrm{d}y$, with $\langle c(\xi)\rangle = \int_0^1 c(\xi, y) \, \mathrm{d}y$. As can be expected from the Laplacian nature of the boundary-value problem, $\sigma^2(\xi) \sim \xi^2$, as confirmed by the analysis of the data, figure 7.

Therefore, the simple case study developed in this section shows that the integral transform theory can be applied to solve partial differential equations in the presence singular non-atomic boundary conditions. Other examples can be worked out within the theoretical framework of integral transform theory, since the solution of many other linear transport and vibrational problems can be expressed in integral form through the introduction of the

corresponding Green's functions.

## 9. Concluding remarks

This paper analysed the properties of integral transforms of singular non-atomic measures associated with linear IFSP. It has been shown that the Fourier–Stieltjes and Laplace–Stieltjes transforms satisfy a functional relation, equations (10) and (27). For unimodular IFSP, a closed-form expression for the integral transforms is obtained in terms of a converging infinite product, equations (16) and (39). Alternatively, the transforms can be evaluated by iterating a recursive scheme, equation (53), and the sequence of approximants is uniformly convergent.

The results obtained enable us to attain a rigorous treatment of integral transforms of singular non-atomic measures, such as multifractal distributions. This allows us to develop a rigorous mathematical physics of distributed parameter systems possessing singular measures.

It is also interesting to observe that equation (10) is the spectral representation of the Markov operator associated with the invariant measure of affine IFSP. It encompasses the entire structure of the moment hierarchy in a single equation. It is expected that this spectral formulation may have some utility in solving the inverse problem of IFSP in connection with signal and image processing and compression.

We have also discussed some physical applications of the theory focusing on optics (scattering, wavelet analysis of complex fractal structures, computations of thermodynamic averages over the density of states) and on the dynamics of heterogeneous systems (adsorption, relaxation), associated with a hierarchical and singular distribution of adsorption energies, relaxation times etc.

To conclude, another important application of the theory should be mentioned, namely the time-series analysis of cascade processes. The results developed in this paper, and in particular the study of Laplace transforms, can be further developed in order to achieve a pointwise characterization of the scaling properties of the underlying cascade process. This approach is particularly interesting in the analysis of experiments and computer simulations of complex phenomena, the underlying dynamics of which can be regarded as a cascade (multiplicative) process, as in the study of mixing (chaotic flows) and turbulence, or in the analysis of dissipative dynamical systems for which the asymptotic behaviour is localized on an invariant subset and the resulting measure displays singularities, as for the period-doubling attractor [52].

## Acknowledgments

The author is grateful to A Adrover, B Forte, M K Schwalm and W A Schwalm for useful discussions.

## References

[1] Berry M V and Percival I C 1986 *Opt. Acta* **33** 577
[2] Allain C and Cloitre M 1986 *Phys. Rev.* B **33** 3566
[3] Dietler G, Aubert C, Cannell D S and Wiltzius P 1986 *Phys. Rev. Lett.* **57** 3117
[4] Vacher R, Woignier P, Pelous J and Courtens E 1988 *Phys. Rev.* B **37** 6500
[5] Feder J, Jossang T and Rosenqvist E 1984 *Phys. Rev. Lett.* **53** 1403
[6] Daoud M, Family F and Jannink J 1984 *J. Physique Lett.* **45** 199

[7] Martin J E and Ackerson B J 1985 *Phys. Rev.* A **31** 1180
[8] Bale H D and Schmidt P W 1984 *Phys. Rev. Lett.* **53** 596
[9] Reed L D 1989 *MS Thesis* Department of Physics, University of North Dakota
[10] Schwalm W A and Schwalm M K 1989 *Phys. Rev.* B **39** 872
[11] Giona M, Schwalm W A, Adrover A and Schwalm M K 1996 *Chem. Engng. J.* **64** 45
[12] Barnsley M F and Demko S 1985 *Proc. R. Soc.* A **399** 243
[13] Forte B and Vrscay E R 1994 *Fractals* **2** 325
[14] Bessis D and Demko S 1991 *Physica* **47D** 427
[15] Abenda S, Demko S and Turchetti G 1992 *Inverse Problems* **8** 737
[16] Abenda S and Turchetti G 1989 *Nuovo Cimento* B **104** 213
[17] Forte B and Vrscay E R 1994 *Appl. Prob. Preprint*
[18] Barnsley M F 1988 *Fractals Everywhere* (Boston, MA: Academic)
[19] Hutchinson J 1981 *Indiana Un. J. Math.* **30** 713
[20] Falconer K 1993 *Fractal Geometry Mathematical Foundations and Applications* (New York: Wiley)
[21] Zygmund A 1968 *Trigonometric Series* vol I (Cambridge: Cambridge University Press) pp 194–7
[22] Schwalm W A and Schwalm M K 1988 *Phys. Rev.* B **37** 9524
[23] Domany E, Alexander S, Bensimon D and Kadanoff L 1983 *Phys. Rev.* B **28** 3110
[24] Hattori K, Hattori T and Watanabe H 1986 *Phys. Lett.* **115A** 207
[25] Hilfer R and Blumen A 1986 *Fractals in Physics* ed L Pietronero and A Tosatti (Amsterdam: Elsevier) p 33
[26] Dominguez D and Wiecko C 1990 *Phys. Rev.* B **41** 9101
[27] Courant R and Hilbert D 1937 *Methods of Mathematical Physics* (Berlin: Springer)
[28] Lohmann A W 1993 *J. Opt. Soc. Am.* **10** 2181
[29] Lohmann A W and Soffer B H 1994 *J. Opt. Soc. Am.* **11** 1798
[30] Hunt J C R, Kevlahan N K-R, Vassilicos J C and Farge M 1993 *Wavelets, Fractals and Fourier Transforms* ed M Farge, J C R Hunt and J C Vassilicos (Oxford: Clarendon) p 1
[31] Arneodo A, Argoul F, Bacry E, Elezgaray J, Muzy J F and Tabard M 1993 *Progress in Wavelet Analysis and Applications* ed Y Meyer and S Roques (Gif-sur Yvette: Éditions Frontières) p 21
[32] Kuhn A, Argoul F, Muzy J F and Arneodo A 1994 *Phys. Rev. Lett.* **73** 2998
[33] Cormen T H, Leiserson C H and Rivest R L 1986 *Introduction to Algorithms* (Cambridge, MA: MIT)
[34] Knuth D E 1973 *The Art of Computer Programming - Vol 3 Sorting and Searching* (Reading, MA: Addison-Wesley)
[35] Wheeler J C and Gordon R G 1970 in *The Padé Approximant in Theoretical Physics* ed G A Baker and J L Gammell (New York: Academic)
[36] Rudzinski W and Everett D H 1992 *Adsorption of Gases on Heterogeneous Surfaces* (London: Academic)
[37] Sircar S and Myers A L 1986 *AIChE J.* **32** 650
[38] Jaroniec M, Madey R and Rothstein D 1987 *Chem. Engng. Sci.* **42** 2135
[39] Ngai K L and Wright G B (ed) 1984 *Relaxations in Complex Systems* (Arlington, DC: Naval Research Laboratory)
[40] Havlin S and Ben-Avraham D 1987 *Adv. Phys.* **36** 695
[41] Dissado L A 1982 *Phys. Scr.* T **1** 110
[42] Nigmatullin R R and Hill R M 1983 *Phys. Status Solidi* b **118** 769
[43] Nigmatullin R R 1984 *Phys. Status Solidi* b **123** 739
[44] Nigmatullin R R 1984 *Phys. Status Solidi* b **124** 389
[45] Nigmatullin R R 1986 *Phys. Status Solidi* b **133** 425
[46] Giona M, Cerbelli S and Roman H E 1992 *Physica* **A191** 449
[47] Rabotnov Yu N 1980 *Elements of Hereditary Solid Mechanics* (Moscow: MIR)
[48] Oldham K B and Spanier J 1974 *The Fractional Calculus* (New York: Academic)
[49] Miller K S and Ross B 1993 *An Introduction to the Fractional Calculus and Fractional Differential Equations* (New York: Wiley)
[50] Giona M and Adrover A 1997 *Fractals and Chaos in Chemical Engineering* ed M Giona and G Biardi (Singapore: World Scientific) in press
[51] Carslaw H S and Jaeger J C 1959 *Conduction of Heat in Solids* (Oxford: Clarendon Press)
[52] Halsey T C, Jensen M H, Kadanoff L P, Procaccia I and Shraiman B I 1986 *Phys. Rev.* A **33** 540